

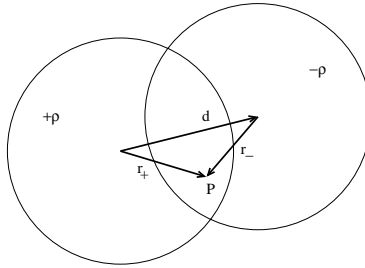
Solution Set 2

1. **Griffiths 2.18** First we need the electric field inside a uniformly charged sphere. Imagining a spherical Gaussian surface of radius r inside the charged sphere, symmetry tells us that the electric field must be pointing radially outward and have the same magnitude over the whole surface. So $\int \mathbf{E}(r) \cdot d\mathbf{a} = 4\pi r^2 E(r) = Q_{\text{enc}}/\epsilon_0 = \rho 4\pi r^3/3$. Thus $\mathbf{E}(r) = \rho r/(3\epsilon_0) \hat{\mathbf{r}} = \rho r/(3\epsilon_0) \mathbf{r}$ where \mathbf{r} is the unit vector from the center of the charged sphere to the point in question.

In this problem we have two charged spheres with the vector \mathbf{d} pointing from the center of the positive sphere to the center of the negative sphere. For a point P in the region of overlap, there will be a contribution to the E-field from both spheres. If we let \mathbf{r}_+ be the vector from the center of the positive sphere to P and \mathbf{r}_- be the vector from the center of the negative sphere to P , then the total E-field at P is

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\epsilon_0} \mathbf{r}_+ + \frac{-\rho}{3\epsilon_0} \mathbf{r}_- = \frac{\rho}{3\epsilon_0} (\mathbf{r}_+ - \mathbf{r}_-).$$

But from the figure, we see that $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So the total electric field in the overlap region is $\mathbf{E} = \rho/3\epsilon_0 \mathbf{d}$.



Problem 1

2. **Griffiths 2.20** We'll calculate the curl of \mathbf{E} , because a real electrostatic field must have zero curl. For (a) we have

$$\nabla \times \mathbf{E} = -2kz \hat{\mathbf{x}} - 3kz \hat{\mathbf{y}} - kx \hat{\mathbf{z}} \neq 0,$$

so (a) is *not* a possible electrostatic field. For (b),

$$\nabla \times \mathbf{E} = k(2z - 2z) \hat{\mathbf{x}} + (0 - 0) \hat{\mathbf{y}} + (2y - 2y) \hat{\mathbf{z}} = \mathbf{0}$$

Thus (b) is a possible electrostatic field. Now we want to compute the potential at some point (x_0, y_0, z_0) , where the origin is at zero potential, using the relationship that $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Let's choose a simple path that goes in straight lines from $(0, 0, 0)$ to $(x_0, 0, 0)$ to $(x_0, y_0, 0)$ to (x_0, y_0, z_0) . There are three parts to the integral, and on each part we have a different $\mathbf{E} \cdot d\mathbf{l} = ky^2 dx + k(2xy + z^2) dy + 2kyz dz$. On the first segment, $y = z = dy = dz = 0$ so we get no contribution because $\mathbf{E} \cdot d\mathbf{l} = 0$. On the second segment, $z = dz = dx = 0$ and $x = x_0$, so we get the contribution $\int \mathbf{E} \cdot d\mathbf{l} = 2kx_0 \int_0^{y_0} y dy = kx_0 y_0^2$. On the final segment, $dx = dy = 0$ while $x = x_0$ and $y = y_0$, so we get $\int \mathbf{E} \cdot d\mathbf{l} = 2ky_0 \int_0^{z_0} z dz = ky_0 z_0^2$. Now replacing x_0 with x and so on, we find $V(x, y, z) = -\int \mathbf{E} \cdot d\mathbf{l} = -k(xy^2 + yz^2)$. We can check that we did the integrals right by computing $\mathbf{E} = -\nabla V = k(y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + 2yz \hat{\mathbf{z}})$, which gives us back the electric field we started with.

3. **Griffiths 2.25 (c)** We can use the second equation in (2.30) to calculate the potential due to a disk with uniform surface charge:

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma(\mathbf{r}')}{r} da'.$$

Using cylindrical coordinates with angle ϕ and radius s , for this case $da' = \sigma s ds d\phi$ and $r = \sqrt{s^2 + z^2}$.

$$V = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^R \frac{\sigma s ds}{\sqrt{s^2 + z^2}} = \frac{2\pi\sigma}{4\pi\epsilon_0} \left[\sqrt{s^2 + z^2} \right]_0^R = \frac{\sigma}{2\epsilon_0} (\sqrt{R^2 + z^2} - z)$$

Now, V is independent of x and y . Thus $\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$, so

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial z} \hat{\mathbf{z}} = -\frac{\sigma}{2\epsilon_0} \left[\frac{1}{2} \frac{2z}{\sqrt{R^2 + z^2}} - 1 \right] = \frac{z\sigma}{2\epsilon_0} \left[\frac{1}{z} - \frac{1}{\sqrt{R^2 + z^2}} \right] \hat{\mathbf{z}}.$$

This is exactly what we got calculating the electric field directly in Problem Set 1.

4. **Griffiths 2.32 (a-b)** First we need to find the potential and electric field produced by a uniformly charged solid sphere of radius R and charge q . Outside the sphere, the electric field looks just like that from a point charge, $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{\mathbf{r}}$. Inside the sphere, we can use Gauss's Law: $\int \mathbf{E} \cdot d\mathbf{a} = 4\pi r^2 E(r) = Q_{\text{enc}}/\epsilon_0 = qr^3/(\epsilon_0 R^3) \Rightarrow E = \frac{1}{4\pi\epsilon_0} \frac{qr}{R^3} \hat{\mathbf{r}}$. To find the potential, we need to do a line integral of the electric field in from infinity. For $r > R$,

$$V(r) = - \int_{\infty}^r \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' = \frac{1}{4\pi\epsilon_0} \frac{q}{r'} \Big|_{\infty}^r = \frac{q}{4\pi\epsilon_0} \frac{1}{r}.$$

For $r < R$,

$$V(r) = - \int_{\infty}^R \frac{1}{4\pi\epsilon_0} \frac{q}{r'^2} dr' - \int_R^r \frac{1}{4\pi\epsilon_0} \frac{q}{R^3} r' dr' = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{R} - \frac{1}{R^3} \left(\frac{r^2 - R^2}{2} \right) \right] = \frac{q}{4\pi\epsilon_0} \frac{1}{2R} \left(3 - \frac{r^2}{R^2} \right)$$

- (a) Using equation 2.43, with $\rho = \frac{q}{(4/3)\pi R^3}$ inside the sphere and zero outside, we have

$$W = \frac{1}{2} \int \rho V d\tau = \frac{1}{2} \frac{3q}{4\pi R^3} \int d\Omega \int_0^R \frac{q}{8\pi\epsilon_0 R} \left(3 - \frac{r^2}{R^2} \right) r^2 dr = \frac{3q^2 4\pi}{64\pi^2 \epsilon_0 R^4} \left[r^3 - \frac{r^5}{5R^2} \right]_0^R = \frac{1}{4\pi\epsilon_0} \left(\frac{3q^2}{5R} \right)$$

- (b) Now using equation 2.45:

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \int E^2 d\tau = \frac{\epsilon_0}{2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 \int d\Omega \left\{ \int_R^{\infty} \frac{1}{r^4} r^2 dr + \int_0^R \frac{r^2}{R^6} r^2 dr \right\} \\ &= \frac{q^2}{8\pi\epsilon_0} \left\{ \left[-\frac{1}{r} \right]_R^{\infty} + \left[\frac{r^5}{5R^6} \right]_0^R \right\} = \frac{1}{4\pi\epsilon_0} \left(\frac{3q^2}{5R} \right) \end{aligned}$$

Fortunately, both the solutions give the same result.

5. Griffiths 2.36 (a-c)

- (a) For each of the cavities, we may imagine a Gaussian surface that is completely in the conductor and surrounds the cavity. Since there is no electric field in the metal of the conductor, $\int \mathbf{E} \cdot d\mathbf{a} = 0$. By Gauss's Law this means that the charge enclosed must be zero, so the total charge on the inner surface of the cavity must be exactly opposite of the point charge contained in the cavity. By symmetry, there is no reason for the surface charge to be anything but uniformly distributed. Thus $\sigma_a = q_a/4\pi a^2$ and $\sigma_b = q_b/4\pi b^2$. Since the conductor is neutral, the charge on the outer surface must be opposite the charge on the inner surface, and again it will be uniformly distributed, so $\sigma_R = (q_a + q_b)/4\pi R^2$.
- (b) To find the field outside the conductor, the argument is exactly the same as in Example 2.9 in the text. The conductor makes the electric field outside look exactly like two point charges q_a and q_b at the origin. So $\mathbf{E} = \frac{1}{4\pi\epsilon_0} \frac{q_a + q_b}{r^2} \hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector from the center of the conducting sphere.

- (c) The surface charge in cavity a cancels the electric field due to the point charge q_a everywhere outside the cavity. So the only source of electric field in cavity a is the point charge q_a and the surface charge. But using Gauss's Law and the spherical symmetry, the electric field inside the cavity is just that of the point charge q_a , namely $\mathbf{E}_a = \frac{1}{4\pi\epsilon_0} \frac{q_a}{r_a^2} \hat{\mathbf{r}}_a$, where \mathbf{r}_a is a unit vector from the center of cavity a . The same reasoning applies to cavity b , so $\mathbf{E}_b = \frac{1}{4\pi\epsilon_0} \frac{q_b}{r_b^2} \hat{\mathbf{r}}_b$.

6. **Griffiths 2.39** To find the capacitance between two conductors, we imagine placing a charge $+Q$ on one and $-Q$ on the other and then calculate the potential difference between them. Then we can find the capacitance from $C = Q/V$. (Note that the capacitance should depend only on the physical size of the system and not on the imaginary charge Q .)

So in this case let's put a charge per unit length $+\lambda$ on the inner cylinder, and $-\lambda$ on the outer cylinder. To calculate the potential difference between the two cylinders, we need to integrate the electric field. Since the charge is evenly distributed, we can draw a Gaussian cylinder with radius r and length L between the two conductors. The electric field is pointing radially, and we can find its magnitude: $\int \mathbf{E} \cdot d\mathbf{a} = 2\pi r L E = Q_{\text{enc}}/\epsilon_0 = \lambda L/\epsilon_0 \Rightarrow E = \lambda/2\pi\epsilon_0 r$. The potential difference is then

$$V(b) - V(a) = - \int_a^b \mathbf{E} \cdot d\ell = - \frac{\lambda}{2\pi\epsilon_0} \int_a^b \frac{1}{r} dr = - \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right).$$

Since the inner cylinder is at a higher potential (the potential drops in going from a to b), the positive voltage between the two conductors is $V = V(a) - V(b) = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$. Then $C = Q/V = \lambda L/V = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$ so the capacitance per unit length is $C/L = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$. Note that this is independent of λ and Q .

7. **Griffiths 2.50** The differential form of Gauss's Law tells us $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$. Thus in this case, we find $\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \epsilon_0 a$. This is a constant, uniform charge density. So why should the electric field point in the x -direction and not in the y -direction? In fact, it could, because you find exactly the same charge density for the fields $\mathbf{E} = ay \hat{\mathbf{y}}$ and $\mathbf{E} = (a/3)\mathbf{r}$. The point is that the *differential equations* $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ and $\nabla \times \mathbf{E} = 0$ are not sufficient to determine the electric field; boundary conditions are also necessary. It is just like asking for a function whose derivative is 3. There are many such possibilities: $f(x) = 3x$, $g(x) = 3x + 10$, $h(y) = 3y + c$; until you know some boundary conditions (such as $f(0) = 2$), you cannot give a unique answer. Knowing the field you can determine the charge distribution, but it doesn't work in reverse: knowing the charge distribution is not always enough to determine the field.
8. **Handout** We have two concentric spherical shells, and the outer one is being driven with a potential $\phi_2(t) = V_0 \cos \omega t$. We make the approximation that the potential between the spheres is $\phi_2(t)$ everywhere. This would be true for the original version of Gauss's Law (i.e. for a zero mass photon), so call that solution the "original solution". The original solution is *not* an exact solution to the new equations, but since the change in the equations is very small, the original solution must be very close to the "new solution". So we will plug in the "original solution" to the new equation and see how much the "original solution" is modified. The error we make here is second order in the difference between the "original solution" and the "new solution", so it can be ignored for the purposes of determining the sensitivity required to carry out this experiment. We start with the given differential equation and integrate both sides over a sphere of radius $R_1 < r < R_2$.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{\phi_2}{\lambda^2} \Rightarrow \int \nabla \cdot \mathbf{E} d\tau = \int \left[\frac{\rho}{\epsilon_0} - \frac{\phi_2}{\lambda^2} \right] d\tau$$

Now we can compute both sides of this equation separately. By symmetry, we assume that the electric field is purely radial and has the same magnitude all over our spherical surface. Thus $\int \mathbf{E} \cdot d\boldsymbol{\tau} = 4\pi r^2 E(r)$. The second term on the right hand side is independent of r , and $\int \rho d\tau = Q_{\text{enc}}$. So we find, after substituting in

the definition of $\bar{\lambda}$,

$$4\pi r^2 E(r) = \left[\frac{Q_{\text{enc}}}{\epsilon_0} - \frac{4}{3}\pi r^3 \frac{\phi_2}{\bar{\lambda}^2} \right] \Rightarrow \mathbf{E}(r) = \left[\frac{q}{4\pi\epsilon_0 r^2} - \frac{V_0 m_0^2 c^2}{3\hbar^2} r \cos \omega t \right] \hat{\mathbf{r}}.$$

Here $Q_{\text{enc}} = q$, the charge on the inner sphere.

Now, to calculate the potential difference we integrate the electric field.

$$V(R_2) - V(R_1) = - \int_{R_1}^{R_2} \mathbf{E} \cdot d\ell = - \int_{R_1}^{R_2} E(r) dr = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) + \frac{V_0 m_0^2 c^2}{3\hbar^2} \cos(\omega t) \frac{1}{2} (R_2^2 - R_1^2)$$

Now lets plug in some numbers. The wording is a little confusing, since V_0 is the amplitude of ϕ_2 , it can't really be the peak-to-peak voltage. So I'll just take $V_0 = 10$ kV and compute the amplitude of the measured voltage. If you did something slightly different, that fine. The other numbers are $q = 0$, $R_1 = 0.5$ m, $R_2 = 1.5$ m, and $m_0 = 10^{-15}$ eV/c². Be careful about converting all the units. I get $\Delta V = 8.53 \times 10^{-14} \cos \omega t$ volts. Anything between half or twice this value is acceptable.